Recall : (X, g^{TX}) oriented compact Riemannian mfd

(E, h^E) Hermitian vector bundle

∇^E Hermitian connection

Generalized Laplacian : $Q \in C^\infty(X, \mathrm{End}(E))$

$Q^* = Q$

$$H = \Delta^E + Q : \underset{\text{Domain}}{\underset{\shortparallel}{W^2(X,E)}} \subset L^2(X,E) \to L^2(X,E)$$

Since H is essentially self-adj, heat operator e^{-tH}, for $t>0$, exists as one-parameter semi-group in $\mathrm{End}(L^2(X,E))$.

Heat kernel $e^{-tH}(x, x')$ or $P_t(x,x')$

$$(e^{-tH}s)(x) = \int_X \underbrace{e^{-tH}(x,x')}_{\text{heat kernel}} s(x')\, dv(x')$$

$\forall\ s \in C^\infty(X,E)$

Note that $\forall\ s \in L^2(X,E)$

$$e^{-tH}s \in L^2(X,E)$$

solves the equation $\left(\frac{\partial}{\partial t} + H\right) e^{-tH}s = 0 \qquad t>0$

$\frac{\partial}{\partial t} + H$ is hypoelliptic

$$\Rightarrow \quad e^{-tH}s \in C^\infty(X,E)$$

$$\Rightarrow \quad e^{-tH},\ t>0, \text{ smoothing operator}$$

Schwartz Kernel Theorem implies

Theorem: There exists a unique heat kernel

$$P_t(x,x') = e^{-tH}(x,x') \in C^\infty(\mathbb{R}_{>0} \times X \times X, E \boxtimes E^*)$$

such that $\left(\frac{\partial}{\partial t}\right)^k P_t(x,x') = (-H_x)^k P_t(x,x')$ $\quad k \in \mathbb{N}$.

Recall: We use the spectral property of H.

Fix ϕ_j ONB of $L^2(X,E)$ s.t.

$$H\phi_j = \lambda_j \phi_j \qquad j = 1, 2, \cdots$$

we know $\phi_j \in C^\infty(X,E)$

Thus:
$$P_t(x,y) = \sum_{j=1}^{+\infty} e^{-t\lambda_j} \phi_j(x) \otimes \underbrace{\phi_j(y)^*}_{\shortparallel \atop h^E(\cdot,\, \phi_j(y))} \qquad (*)$$

proof: ① $\sum_j e^{-t\lambda_j} < +\infty$

$$e^{-t\lambda_j} = \langle e^{-tH}\phi_j, \phi_j \rangle_{L^2}$$

$$\sum_j e^{-t\lambda_j} = \sum_j \| e^{-\frac{tH}{2}} \phi_j \|^2_{L^2}$$

$$= \sum_j \int_X \left| \int_X P_{\frac{t}{2}}(x,y) \phi_j(y)\, dv(y) \right|^2_{h^E_x} dv(x)$$

$$= \int_X \sum_j \left| \langle P_{\frac{t}{2}}(x,\cdot), \phi_j(\cdot) \rangle_{L^2} \right|^2 dv(x)$$

$$= \int_{X\times X} |P_{\frac{t}{2}}(x,y)|^2_{h^E_x \otimes h^{E^*}_y} \, d\nu(x)\, d\nu(y)$$ ③

$$< +\infty$$

C^∞ in (x,y)

② Now we need to prove that the series $(*)$ is convergent in any C^ℓ-norm on $\mathbb{R}_{\geq 0}\times X\times X$

$1 \leq N \leq N'$

$$Q^t_{N,N'}(x,y) := \sum_{j=N}^{N'} e^{-t\lambda_j} \phi_j(x) \otimes \phi_j(y)^*$$

For any $k_1, k_2, k_3 \in \mathbb{N}$

$$\| H_x^{k_1} H_y^{k_2} \left(\frac{\partial}{\partial t}\right)^{k_3} Q^t_{N,N'}(x,y) \|_{L^2}$$

$$= \| \sum_{j=N}^{N'} \lambda_j^{k_1+k_2+k_3} e^{-t\lambda_j} \phi_j(x) \otimes \phi_j(y)^* \|_{L^2}$$

$$= \sum_{j=N}^{N'} \lambda_j^{2k_1+2k_2+2k_3} e^{-2t\lambda_j}$$

$$C = \sup_{s>0} s^{2k_1+2k_2+2k_3} e^{-s}$$

$$\leq C_{k_1,k_2,k_3} t^{-2k_1-2k_2-2k_3} \sum_{j=N}^{N'} e^{-t\lambda_j} \longrightarrow 0$$

as $N \to +\infty$, $N' \to +\infty$

$$\Rightarrow \quad Q^t_{1,N'}(x,y) \longrightarrow Q^t_{1,\infty}(x,y) \text{ in } C^\infty\text{-top in } (t,x,y)$$

At the same time

$$\left(\frac{\partial}{\partial t} + H_x\right) Q^t_{1,\infty}(x,y) = \lim_{N'\to+\infty} \left(\frac{\partial}{\partial t} + H_x\right) Q^t_{1,N'}(x,y)$$

$$= 0$$

&

$$(Q^t_{1,\infty} s)(x) = \int_X Q^t_{1,\infty}(x,y) s(y) dv(y)$$

$$\| Q^t_{1,\infty} s - s \|_{L^2} \le \| Q^t_{1,N'} s - Q^0_{1,N'} s \|_{L^2} + 2 \| Q^0_{N',\infty} s \|_{L^2}$$

As $t \to 0$

$$\| Q^t_{1,N'} s - Q^0_{1,N'} s \|_{L^2} \to 0$$

Provided $\| Q^0_{N',+\infty} s \|_{L^2} \le$ any given ε

$\Rightarrow$ $Q^t_{1,\infty} s \longrightarrow s$ in $L^2(X, E)$ as $t \to 0$.

Uniqueness of heat kernel $\Rightarrow$ $P_t(x,y) = Q^t_{1,\infty}(x,y)$

#

Def (Hilbert-Schmidt & trace class)

$\mathcal{H}$ separable Hilbert space

① $A \in \mathcal{L}(\mathcal{H})$ is Hilbert-Schmidt (HS) if $\forall \{e_j\}$ ONB of $\mathcal{H}$

$$\|A\|^2_{HS} := \sum_{j,k} |\langle A e_j, e_k \rangle_{\mathcal{H}}|^2 < \infty .$$

② $A \in \mathcal{L}(H)$ is trace class if

$A = A_1 \circ A_2$ for A_1, A_2 HS

$$\mathrm{Tr}[A] := \sum_i \langle A e_i, e_i \rangle_{\mathcal{H}}$$

Prop: If $K : L^2(X,E) \to L^2(X,E)$ has a smooth integral kernel, then K is HS.

Prop: Heat operator e^{-tH}, $t>0$, is trace class. Moreover

$$\mathrm{Tr}[e^{-tH}] = \int_X P_t(x,x)\, dv(x)$$

Proof: ① $e^{-tH} = \underbrace{e^{-\frac{tH}{2}} \circ e^{-\frac{tH}{2}}}_{HS}$ trace class

② $H\phi_j = \lambda_j \phi_j$

$$\mathrm{Tr}[e^{-tH}] = \sum_j e^{-t\lambda_j} = \int_X \sum_j e^{-t\lambda_j} |\phi_j(x)|^2 dv(x)$$

$$= \int_X P_t(x,x)\, dv(x)$$

#

VI.3 McKean-Singer theorem

(X, g^{TX}) cpt, oriented, even-dim

(E, ∇^E, h^E) $\mathbb{Z}_2$-graded Clifford module

Dirac op.

$$D = \sum_{j=1}^m c(e_j) \nabla^E_j : \mathrm{Dom}(D) = W^1 \subset L^2(X,E) \to L^2(X,E)$$

essentially self-adjoint.

Since D is elliptic, so is D^2

$$\mathrm{Spec}(D^2) = \{0 \le \lambda_1 < \lambda_2 \cdots \to +\infty\}$$

each λ_j is eigenvalues with finite multiplicities n_j

In particular $\ker D = \ker D^2$.

$$D^E_\pm : \mathrm{Dom}(D_\pm) \subset W^1(X, E^\pm) \to L^2(X, E^\mp)$$

<u>Cor</u>: $\ker D_+$, $\mathrm{coker}\, D_+ \simeq \ker D_-$ are finite dimensional, so ~~that~~

$\mathrm{Ind}(D_+)$ is well-defined.

Thm (McKean-Singer) For $t > 0$, we have

$$\mathrm{Ind}(D_+) = \mathrm{Tr}_s[e^{-tD^2}]$$

where Tr_s is taken w.r.t. $C^\infty(X, E^\pm)$

Pf: 1st proof:

Note that
$$\ker D^2 \cap C^\infty(X, E^+) = \ker D_+$$
$$\ker D^2 \cap C^\infty(X, E^-) = \ker D_- \cong \mathrm{coker}\, D_+$$

So at first we have

$$\mathrm{Ind}(D_+) = \mathrm{Tr}_s[\,P_{\ker D^2}\,]$$

Projection map $L^2(X,E) \xrightarrow{\perp} \ker D^2$

$$= \lim_{t\to+\infty} \mathrm{Tr}_s[\exp(-tD^2)]$$

<u>Claim</u>: $\mathrm{Tr}_s[\exp(-tD^2)]$ is independent of $t>0$.

$$\frac{\partial}{\partial t}\mathrm{Tr}_s[\exp(-tD^2)] = -\mathrm{Tr}_s[\,D^2\exp(-tD^2)\,]$$

$$= -\frac{1}{2}\mathrm{Tr}_s\left[\left[D\exp(-\tfrac{t}{2}D^2),\ D\exp(-\tfrac{t}{2}D^2)\right]\right]$$

$$= 0$$

$D\exp(-\frac{t}{2}D^2)$ is an odd op. with smooth integral kernel trace class

Tr_s vanishes on $[\,,\,]$.

X cpt

<u>HW</u>: If K_1, K_2 trace class operators on $L^2(X,E)$ with smooth integral kernels $k_1(x,y)$, $k_2(x,y)$

Then $\mathrm{Tr}_s[[K_1,K_2]] = 0$.

2nd Proof:

Set $H_\lambda^\pm := \{ s\in C^\infty(X,E^\pm) : D^2 s = \lambda s\}$

$n_\lambda^\pm := \dim H_\lambda^\pm$

Then $L^2(X,E^\pm) = \bigoplus_{\lambda\in \mathrm{Spec}(D^2)} H_\lambda^\pm$

For $t>0$

$$\mathrm{Tr}_s[\exp(-tD^2)] = \sum_{\lambda\geq 0}(n_\lambda^+ e^{-\lambda t} - n_\lambda^- e^{-\lambda t})$$

Moreover, for $s \in H_\lambda^+$

$$D^2 s = \lambda s$$

$$D^2 D s = D(D^2 s) = \lambda D s$$

$$\Rightarrow D s \in H_\lambda^-$$

If $\lambda \neq 0$ $\quad D : H_\lambda^+ \xrightarrow{\sim} H_\lambda^-$

$$n_\lambda^+ = n_\lambda^-$$

$$\Rightarrow \quad \mathrm{Tr}_s[\exp(-tD^2)] = n_0^+ - n_0^- = \mathrm{Ind}(D_+)$$

#

Thm (heat kernel expansion)

$\exists \; a_j \in C^\infty(X, \mathrm{End}(E))$ s.t. $\forall N$

$$(*) \qquad P_t(x,x) = t^{-\frac{m}{2}} \sum_{j=0}^{N} a_j(x) t^j + O(t^{N+1-\frac{m}{2}})$$

uniformly for $t \in (0, \frac{1}{2}]$

$x \in X$

<u>Rk</u>: On $\mathbb{R}^m$, $\quad D^2 = -\sum_j \frac{\partial^2}{\partial x_j^2}$

$$P_t(x,y) = \frac{1}{(4\pi t)^{m/2}} e^{-\frac{\|x-y\|^2}{4t}}$$

$$P_t(x,x) = \frac{1}{(4\pi)^{m/2}} t^{-m/2}$$

In general, $P_t(x,y)$ on $X \times X$ can be constructed locally as an asymptotic formula.

By McKean-Singer thm and (*), we obtain

$$\operatorname{Ind}(D_+) = \sum_{j=0}^{N} t^{j-\frac{m}{2}} \int_X a_j(x)\, d\nu(x) + O(t^{N+1-\frac{m}{2}})$$

$t < 1$

$m = \dim X$
$= \text{even}$

As a consequence

$$\int_X a_j(x)\, d\nu(x) = \begin{cases} 0 & \text{if } j \neq \frac{m}{2} \\ \operatorname{Ind}(D_+) & \text{if } j = \frac{m}{2} \end{cases}$$

Thm (local index theorem)

$$\operatorname{Tr}_s^E [a_j(x)]\, d\nu(x) = \begin{cases} 0 & j < \frac{m}{2} \\ [\hat{A}(TX, \nabla^{TX})\, ch^{E/S}(E, \nabla^E)]^{max} & j = \frac{m}{2} \end{cases}$$

top degree component